Church Numerals

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Objectives
You should be able to...

- Explain the form of a Church numeral
- Define some operations on Church numerals: inc, plus, times
- Explain how to represent boolean operations: and, or, not, if
What is a Number?

- The Lambda Calculus doesn’t have numbers.
- A number \( n \) can be thought of as a potential: someday we are going to do something \( n \) times.

Some Church Numerals

1. \( f_0 = \lambda f \rightarrow \lambda x \rightarrow x \)
2. \( f_1 = \lambda f \rightarrow \lambda x \rightarrow f \, x \)
3. \( f_2 = \lambda f \rightarrow \lambda x \rightarrow f \, (f \, x) \)
4. \( f_3 = \lambda f \rightarrow \lambda x \rightarrow f \, (f \, (f \, x)) \)

Prelude> let show m = m (+1) 0
Prelude> show (\f x -> f (f x))
2
Incrementing Church Numerals, 0

- To increment a Church Numeral, what do we want to do?

Running Example

1 \texttt{finc }= \texttt{undefined}
Introduction

Church Numerals

Incrementing Church Numerals, 1

- To increment a Church Numeral, what do we want to do?
- First step, take the church numeral you want to increment.

Running Example

```latex
finc = \lambda m \to undefined
```
To increment a Church Numeral, what do we want to do?
First step, take the church numeral you want to increment.
Second step, return a church numeral representing your result.

Running Example

\[
\text{finc} = \lambda m \rightarrow \lambda f \ x \rightarrow \text{undefined}
\]
Incrementing Church Numerals, 3

- To increment a Church Numeral, what do we want to do?
- First step, take the church numeral you want to increment.
- Second step, return a church numeral representing your result.
- Third step, apply $f$ to $x$, $m$ times.

Running Example

1. $\text{finc} = \lambda m \rightarrow \lambda f \ x \rightarrow m \ f \ x$
Incrementing Church Numerals, 4

- To increment a Church Numeral, what do we want to do?
- First step, take the church numeral you want to increment.
- Second step, return a church numeral representing your result.
- Third step, apply \( f \) to \( x \), \( m \) times.
- Finally, apply \( f \) once more to the result.

Running Example

\[
finc = \lambda m \rightarrow \lambda f \; x \rightarrow f \; (m \; f \; x)
\]
Adding Church Numerals

- Similar reasoning can yield addition and multiplication.
- Here is addition. Can you figure our multiplication? Hint: What does \((nf)\) do?

Running Example

1 fadd m n = \f x -> m f (n f x)

- In your activity you will think about how to represent booleans.
- If a numeral represents a potential number of actions, what would a boolean represent?